On Domination in Fuzzy Graphs

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Abstract - A set $D \subset V$ of a given fuzzy graph $G(V, \rho, \mu)$ is a dominating set if for every $u \in V - D$ there exists $v \in D$ such that (u, v) is a strong arc and $\rho(u) \leq \rho(v)$ and if the number of vertices of D is minimum then it is called a minimum dominating set of G. Domination number of G is the sum of membership values of vertices ofa minimum dominating setD and it is denoted by $\gamma(D)$. In this paper we study domination in fuzzy graphs. Also we formulate an algorithm to find dominating set for a given fuzzy graph.

Keywords: fuzzy domination, strong arc, weak arc, strength of connectedness.

I. INTRODUCTION

It is quite well known that graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a Fuzzy Graph Model.

Application of fuzzy relations are widespread and important; especially in the field of clustering analysis, neural networks, computer networks, pattern recognition, decision making and expert systems. In each of these, the basic mathematical structure is that of a fuzzy graph.

Harary et al[2] explained an interesting application in voting situations using the concept of domination in graphs. Rosenfeld[4] introduced the notion of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as paths, cycles and connectedness. Somasundaram A. and Somasundaram S. [5] discussed about domination in fuzzy graph. The concept of strong arcs was introduced by Bhutani[1] in which an arc of a fuzzy graph is strong if and only if its weight is equal to the strength of connectedness of its end nodes. In this paper we define dominating set in a fuzzy graph $G(V, \rho, \mu)$ connecting the concepts of strong arcs and membership values of vertices of G.

II. PRELIMINARIES

Let us see the following basic definitions in a fuzzy graph. [3].
Definition 1: A *fuzzy subset* of a nonempty set V is a mapping $\sigma: V \longrightarrow [0,1]$.
Definition 2: A *fuzzy relation* on V is a fuzzy subset of V × V.
Definition 3: A *fuzzy graph* $G = (\sigma, \mu)$ is a pair of function $\sigma: V \to [0,1]$ and
$\mu: V \times V \to [0,1]$ where $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$.

Definition4: The underlying crisp graph of $G = (\sigma, \mu)$ is denoted by $G^* = (V, E)$ where
$V = \{u \in V : \sigma(u) > 0\}$ and
$E = \{(u, v) \in V \times V : \mu(u, v) > 0\}$.

Definition5: The *order p* and *sizeq* of the fuzzy graph $G = (\sigma, \mu)$ are defined by
$p = \sum_{u \in V} \sigma(v)$ and $q = \sum_{(u,v) \in E} \mu(u, v)$.

Definition6: Let G be a fuzzy graph on V and $S \subseteq V$, then the *fuzzy cardinality* of S is defined to be $\sum_{u \in S} \sigma(u)$.

Definition 7: The *strength of the connectedness* between two nodes u, v in a fuzzy graph G is $\mu^{\infty}(u, v) = sup\{\mu^k(u, v) : k = 1,2,3..\}$ where $\mu^k(u, v) = sup \left\{ \begin{array}{c} \mu(u, u_1) \wedge \mu(u_1, u_2) \wedge \mu(u_2, u_3) \\ \wedge \mu(u_3, u_4) \wedge .... \mu(u_{k-1}, v) \end{array} \right\}$.

Definition8: An arc (u, v) is said to be a *strong arc* if $\mu(u, v) = \mu^{\infty}(u, v)$ If $\mu(u, v) = 0$ for every $v \in V$, then u is called an *isolated node*.

Let us now define dominating set, minimum dominating set and domination number in a fuzzy graph

Definition 9: Let $G(V, \rho, \mu)$ be a fuzzy graph and $D \sqsubseteq V$, D is a *dominating set* if for every $u \in V - S$ there exist $v \in D$ such that (i) (u, v) is a strong arc and (ii)$\rho(u) \leq \rho(v)$.
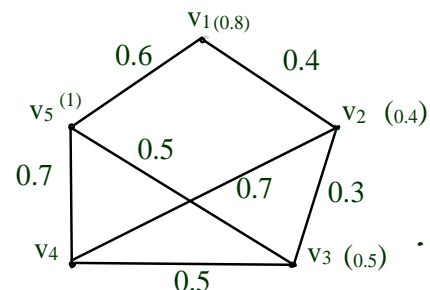

Figure 1 D = {V₂, V₅}

Definition 10: A dominating set of a fuzzy graph with minimum number of vertices is called a *minimum dominating set*.
Definition 11:*Domination number* of a fuzzy graph is the sum of membership values of the vertices of a minimum dominating set.

III. MAIN RESULTS

Proposition1: A fuzzy graph remains connected even after removal of all weak edges in it.

Proof:

Let G be a fuzzy graph and $\mu^{\infty}(u,v)$ denotes the strength of connectedness between two vertices u and v. Let e be a weak edge in G and $G' = G - e$.

To prove: G' is connected.

Let us prove this by method of contradiction. Assume that G' is a disconnected graph. Suppose that the weak edge $e = (u,v)$ makes the graph G' disconnected into more than one component. That implies that there is no path between u and v except the edge $e = (u,v)$ in G.Therefore $\mu(u,v) = \mu^{\infty}(u,v)$ which is a contradiction.□

An Algorithm to find Dominating set

Step 1: Find $\mu^{\infty}(u,v)$ for all edges (u,v)

Step 2: Delete all the weak edges

Step 3: Select the vertex u with maximum ρ (u) in G'

Step 4: Group the vertices dominated by u, as V_1

Step 5: Find $G' - V_1$

Step 6: Repeat the steps 3 to 5 until we get isolated vertices.

Step 7: The collection of vertices selected in step 3 and the isolated vertices will form a dominating set.
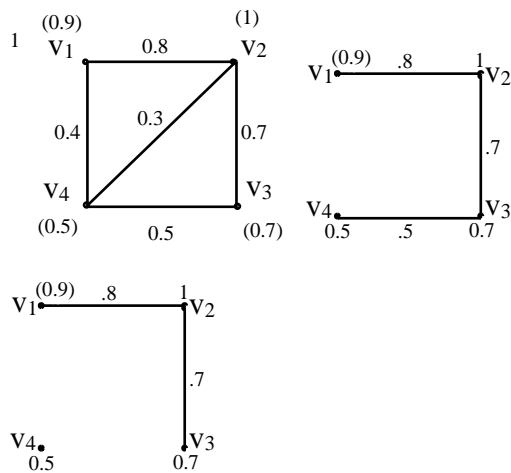
IV. AN ILLUSTRATION

Figure 2 different steps of the above algorithm

In the above Fuzzy graph the dominating set is $\mathbf{D} = \{\mathbf{v_2}, \mathbf{v_4}\}$.

Proposition 2: In a fuzzy cycle all arcs are strong arcs.

Proof: Let G be a fuzzy cycle. Assume that all arcs are not strong arcs.Then there exist at least one weak edge in G. Let us discuss the proof in two cases.

Case 1: Only one weak edge exists.The membership value of the edgeis minimum.

But in fuzzy cycle there should be at least two edges with minimum membership valuewhich is a contradictionto our assumption.

Case 2:more than two edges with minimum membership value. Then the edges will not be weak edges.□

Proposition 3: Let D be a dominating set of a fuzzy graph G.Then for each $\boldsymbol{v} \in \boldsymbol{D}$, there is no vertex in D dominates v.

Proposition 4:Let D be a dominating set of a fuzzy graph G then no bridge exists between any two vertices of $\boldsymbol{V} - \boldsymbol{D}$.
Proof: Suppose there exists a bridge between any two $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V} - \boldsymbol{D}$.Then (u,v) is a strong arc. Then either u dominates v or v dominates u or v dominates u.Therefore either $\boldsymbol{u} \in \boldsymbol{D}$ or $\boldsymbol{v} \in \boldsymbol{D}$.□

Proposition 5: Let D be aminimum dominating set of a fuzzy graph G .If $|\boldsymbol{D}| \geq \left|\frac{V}{2}\right|$, then $\boldsymbol{\gamma}(\boldsymbol{D}) \geq \boldsymbol{\gamma}(\boldsymbol{V} - \boldsymbol{D})$.

Proof: Let us assume $\boldsymbol{\gamma}(\boldsymbol{D}) < \boldsymbol{\gamma}(\boldsymbol{V} - \boldsymbol{D})$ .Then number of vertices in D must be less than number of vertices in $V - D$.Therefore $|D| < |V - D|$.But $|D| + |V - D| = |V|$, $|V - D| = |V| - |D| > |D|, |V| > 2|D|$ .Then $|D| < \left|\frac{V}{2}\right|$□

Proposition 6: If $\gamma(D) < \gamma(V - D)$,then there exists atleast one $v \in D$ dominates two or more vertices of $V - D$, where D is a minimum dominating set.

Proof: Suppose no vertices of D dominates two or more vertices of $V - D$ .Then every vertices in D dominates a vertex in $V - D$, which implies $|D| = |V - D|$ .This implies that.$\gamma(D) \geq \gamma(V - D)$ by definition of domination□
It is easy to prove the following property.

Proposition 7: Let D be a dominating set of a fuzzy graph G and D' is a dominating set of the supp G,then $|D| \geq |D'|$.

V. CONCLUSION

In this paper we definea dominating set, minimum dominating set and domination number in a fuzzy graph and an algorithm is also formulatedfor finding adominating set of a fuzzy graph. Various results regarding the domination number of a fuzzy graph are discussed.

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