Degree Distance of Some Planar Graphs

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Abstract - A novel graph invariant: degree distance index was introduced by Andrey A. Dobrynin and Amide A. Kochetova [3] and at the same time by Gutman as true Schultz index [4]. In this paper we establish explicit formulae to calculate the degree distance of some planar graphs.

Index terms— Degree distance, planar graph, tree graph, regular graph, star graph.

I. INTRODUCTION

Topological indices are numbers associated with molecular graphs for the purpose of allowing quantitative structure-activity/property/toxicity relationships. Wiener index is the best known topological index introduced by Harry Wiener to study the boiling points of alkane molecules. By now there exist a lot of different types of such indices (based either on the vertex adjacency relationship or on the topological distances) which capture different aspects of the molecular graphs associated with the molecules under study. For a graph \( G = (V, E) \) the degree distance of \( G \) is defined as

\[
DD(G) = \sum_{\{u,v\}\subset V(G)} (\deg_G(u) + \deg_G(v))d_G(u,v)
\]

where \( \deg_G(u) \) is the degree of the vertex \( u \) in \( G \) and \( d_G(u,v) \) is the shortest distance between \( u \) and \( v \). Klein et al. [5] showed that if \( G \) is a tree on \( n \) vertices then \( DD(G) = 4W(G) - n(n-1) \) where \( W(G) = \sum_{\{u,v\}\subset V(G)} d_G(u,v) \) is the Wiener index of the graph \( G \). Thus the study of the degree distance for trees is equivalent to the study of the Wiener index. But the degree distance of a graph is a more sensitive invariant than the Wiener index.

In this paper we compute the degree distance of some planar and tree graphs. In Section II, we introduce multi-star graph \( K_{1,n,n,n} \) [2] and calculate its degree distance, in Section III, we calculate the degree distance of planar graphs \( P_n \) (\( n \geq 5 \)) and \( P_m \times C_n \) (\( m \geq 2, n \geq 3 \)) and in Section IV, We introduce the graph operation \( \hat{\circ} \) and compute the degree distance of \( (P_m \times C_n) \hat{\circ} (P_m \times C_n) \).

II. MULTI-STAR GRAPH

\( K_{1,n,n,n} \) is a graph obtained by introducing an edge to each of the pendant vertices of the star graph and \( K_{2,2} \) is a multi-star graph obtained by introducing an edge to each of the pendant vertices of the star graph

2.1 Construction

Starting from the star graph \( K_{1,n} \) with vertices \( \{v_0, v_1, v_2, \ldots, v_n\} \) introduce an edge to each of the pendant vertices \( v_1, v_2, \ldots, v_n \) to get the resulting graph \( K_{1,n,n} \) with vertices \( \{v_0, v_1, \ldots, v_n, v_{(n+1)}, \ldots, v_{2n}\} \), again introduce an edge to each of the pendant vertices \( v_{(n+1)}, \ldots, v_{2n} \) to get the graph \( K_{1,n,n,n} \).

Repeating this procedure \( (m-1) \) times the resulting graph \( K_{1,n,n,n} \) with \( (mn+1) \) vertices

\[
\{v_0, v_1, v_2, \ldots, v_n, v_{(n+1)}, \ldots, v_{2n}, v_{3n}, \ldots, v_{(m-1)n+1}, \ldots, v_{mn}\}
\]

and \( mn \) edges is obtained as shown in Fig. 1.

![Fig. 1. Multi-star graph \( K_{1,n,n,n} \).](image-url)
Then, \[ DD\left( K_{n,n,n,...,n} \right) \]
\[ = \sum_{i<j} \left( \deg(v_i) \times \deg(v_j) \right) d(v_i, v_j) \]
\[ = \left[ n(n+2)(1+2+...+(m-1))+n(n+1)m \right] + \]
\[ \left[ 4(2 \times 1 + 2 \times 2 +...+2 \times (m-1))((n-1)+(n-2)+...+1) \right] + \]
\[ \left[ 4n(n-1)[2 \times 1 + 2 \times 2 +...+2 \times (m-2) +1] \right] + \]
\[ \left[ 3n(n-1)[2 \times 1 + (m-1)+2 \times 2 +...+(m-2)+1] \right] + \]
\[ \left[ 4n(1+2+...+(m-2)) + 4n(1+2+...+(m-3)) +...+4n(1) \right] + \]
\[ \left[ 3n((m-1)+(m-2)+...+1) + 2 \times 2m \times ((n-1)+...+1) \right] \]
\[ = \frac{nm}{3} \left( 6m^2n - 4m^2 + 3mn + 1 \right) \]

III. PLANAR GRAPHS

In this section we compute the degree distance of planar graph with maximal edges \[ P_{mn} \] (Fig. 2) and structured web graph \[ P_{m} \times C_{n} (m \geq 2, n \geq 3) \] (Fig. 3).

Result 3.1: Degree distance of \[ P_{n} \] (\( n \geq 5 \)) graph is

\[ DD(P_{n}) = 10n^2 - 48n + 68 \]

![Fig. 2. \( P_{n} \) (\( n \geq 5 \)) graph.](image)

Prove: Let the \( n \) vertices of the \( P_{n} \) (\( n \geq 5 \)) graph be \( \{u_{1}, u_{2},...,u_{n}\} \). Then the degree distance of \( P_{n} \) is

\[ DD(P_{n}) = \sum_{i<j} \left( \deg(u_i) + \deg(u_j) \right) d(u_i, u_j) \]. Note that

\[ \deg(u_1) = \deg(u_n) = n-1; \deg(u_2) = \deg(u_{n-1}) = 3; \]
\[ \deg(u_3) =... = \deg(u_{n-2}) = 4; d_G(u_1, u_j) = 1, j = 2,...,n; \]
\[ d_G(u_i, u_j) = 1, j = 1,...,n, j \neq 2; d_G(u_j, u_{j+1}) = 1, j = 3,...,n-1; \]
\[ d_G(u_i, u_j) = 2, i = 3,4,...,(n-2); (i+2) \leq j \leq n \].

\[ DD(P_{n}) \]
\[ = n(n-1)+2(1+4(n-4)) + \left[ (n-2)(n-1)+2(3)+4(n-4) \right] \]
\[ + \left[ 4n(1+2+...+(m-2)) + 4n(1+2+...+(m-3)) +...+4n(1) \right] + \]
\[ \left[ 41+2+...+(m-2) \right] + 4 \left[ 1+2+...+(m-3) \right] +...+4 \left[ 1 \right] + \]
\[ 3 \left[ 1+2+...+(m-1) \right] + 3 \left[ 1+2+...+(m-2) \right] +...+3 \left[ 1 \right] \]
\[ = 10n^2 - 48n + 68 \]

Result 3.2: Degree distance of structured web 3-regular graph \[ P_{m} \times C_{n} (m \geq 2, n \geq 3) \] is

\[ DD(P_{m} \times C_{n}) = \begin{cases} \frac{3}{4} m^2n(n^2-1) + mn^2 \left( m^2-1 \right) & \text{when } n \text{ is odd} \\ \frac{3}{4} m^2n + mn^2 \left( m^2-1 \right) & \text{when } n \text{ is even} \end{cases} \]

![Fig. 3. \( P_{m} \times C_{3} \) graph.](image)

Prove: Case (i) when \( n \) is odd.

\[ DD(P_{m} \times C_{n}) = \sum_{i<j} \left( \deg(u_i) + \deg(u_j) \right) d(u_i, u_j) \]
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**IV. NEW GRAPH OPERATION \( \hat{e} \)**

In this section we introduce new graph operation \( \hat{e} \) and establish the degree distance of \( (P_m \times C_n) \hat{e} (P_{m_2} \times C_{n_2}) \).

**Definition 4.1:** \( G_1 \hat{e} G_2 \) is a connected graph obtained from \( G_1 \) and \( G_2 \) by introducing an edge \( e \) between an arbitrary vertex of \( G_1 \) and an arbitrary vertex of \( G_2 \).

If \( G_1 \{ p_1, q_1 \} \) has \( p_1 \) vertices and \( q_1 \) edges and \( G_2 \{ p_2, q_2 \} \) has \( p_2 \) vertices and \( q_2 \) edges then \( G_1 \hat{e} G_2 \) will have \( (p_1 + p_2) \) vertices and \( (q_1 + q_2 + 1) \) edges. If \( G_1 = P_m \times C_n \) and \( G_2 = P_{m_2} \times C_{n_2} \) an interesting graph structure \( G = G_1 \hat{e} G_2 \) (example shown in Fig. 4) is obtained by using the new graph operation defined above and we prove the result.

**Result 4.2:** Degree distance of \( G = (P_m \times C_n) \hat{e} (P_{m_2} \times C_{n_2}) \) is

\[
DD(G) = DD(G_1) + DD(G_2) + (6n_2 + 2)D_{G_1}((u_i, u_j))
+ (6n_1 + 2)D_{G_2}((v_i, v_j))
+ n_1 + n_2 + 6n_1n_2
\]

**Proof:** Let an edge be introduced between the arbitrary vertex \( u_i \) of \( G_1 \) and the arbitrary vertex \( v_j \) of \( G_2 \), then the degree distance of \( G = G_1 \hat{e} G_2 = (P_m \times C_n) \hat{e} (P_{m_2} \times C_{n_2}) \) is given by

\[
DD(G) = \sum_{i,j} (\deg_{G_1}(u_i) + \deg_{G_2}(v_j))d_{G_1}(u_i, u_j)
+ \sum_{i,j} (\deg_{G_1}(u_i) + \deg_{G_2}(v_j))d_{G_2}(v_j, v_j)
+ \sum_{i,j} (\deg_{G_1}(u_i) + \deg_{G_2}(v_j))d_{G_1}(v_j, v_j)

= DD(G_1) + \sum_{i,j} d_{G_1}(u_i, u_j) + DD(G_2) + \sum_{i,j} d_{G_2}(v_j, v_j)
+ 6n_2 \sum_{i,j} d_{G_1}(u_i, u_j) + 6n_1n_2
+ \sum_{i} d_{G_1}(u_i, u_i) + 6n_1n_2
+ \sum_{i,j} d_{G_2}(v_j, v_j) + 6n_2 \sum_{i,j} d_{G_2}(v_j, v_j)

= DD(G_1) + DD(G_2) + (6n_2 + 2)D_{G_1}((u_i, u_j))
+ (6n_1 + 2)D_{G_2}((v_i, v_j))
+ n_1 + n_2 + 6n_1n_2
\]
To find $D_{G_1}(u_i, u_j)$ when $n_1$ is even,
\[
D_{G_1}(u_i, u_j) = \sum_{r \in \mathcal{V}(G_1)} d_{G_1}(u_i, u_j) \\
= \left[ 2 \left(1 + 2 + \ldots + \left(\frac{n_1 - 1}{2}\right)\right) \right] \\
+ \left[ 2 \left((1+1) + (2+1) + \ldots + \left(\frac{n_1 - 1}{2}+1\right)\right)\right] + 1 \\
+ \left[ 2 \left((1+2) + (2+2) + \ldots + \left(\frac{n_1 - 1}{2}+2\right)\right)\right] + 2 \\
+ \ldots + \left[ 2 \left((1+(m-1)) + (2+m-1)\right)\right] + (m_1-1) \\
= \frac{n_1 m_1 (m_1-1)}{2} + m_1 n_1^2 \\
\]

Similarly,
\[
D_{G_1} (v_i, v_j) = \sum_{r \in \mathcal{V}(G_1)} d_{G_1} (v_i, v_j) \\
= \frac{n_2 m_2 (m_2-1)}{2} + m_2 n_2^2 \\
\]

To find $D_{G_1}(u_i, u_j)$ when $n_1$ is odd,
\[
D_{G_1}(u_i, u_j) = \sum_{r \in \mathcal{V}(G_1)} d_{G_1}(u_i, u_j) \\
= \left[ 2 \left(1 + 2 + \ldots + \left(\frac{n_1 - 1}{2}\right)\right) \right] \\
+ \left[ 2 \left((1+1) + (2+1) + \ldots + \left(\frac{n_1 - 1}{2}+1\right)\right)\right] + 1 \\
+ \left[ 2 \left((1+2) + (2+2) + \ldots + \left(\frac{n_1 - 1}{2}+2\right)\right)\right] + 2 \\
+ \ldots + \left[ 2 \left((1+(m-1)) + (2+m-1)\right)\right] + (m_1-1) \\
= \frac{n_1 m_1 (m_1-1)}{2} + m_1 n_1^2 \\
\]

Similarly,
\[
D_{G_1} (v_i, v_j) = \sum_{r \in \mathcal{V}(G_1)} d_{G_1} (v_i, v_j) \\
= \frac{n_2 m_2 (m_2-1)}{2} + m_2 n_2^2 \\
\]

Using Result 3.2,
\[
DD(G_1) = DD(P_n \times C_n) = \begin{cases} 
\frac{3}{4} n_1^2 m_1 (m_1^2 - 1) + m_1 n_1 (m_1^2 - 1) & \text{when } n_1 \text{ is odd} \\
\frac{3}{4} n_1^2 m_1 + m_1 n_1 (m_1^2 - 1) & \text{when } n_1 \text{ is even} 
\end{cases} \\
DD(G_2) = DD(P_n \times C_n) = \begin{cases} 
\frac{3}{4} n_2^2 m_2 (m_2^2 - 1) + m_2 n_2 (m_2^2 - 1) & \text{when } n_2 \text{ is odd} \\
\frac{3}{4} n_2^2 m_2 + m_2 n_2 (m_2^2 - 1) & \text{when } n_2 \text{ is even} 
\end{cases} 
\]