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Power Domination in WK - Recursive Networks

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Abstract – Electric power networks mustbecontinuouslymonitored. Suchmonitoring can beefficiently accomplished by placing phase measurement units (PMUs) at selected network locaions.DuetothehighcostofthePMUs,their number must be minimized. The problem of monitoringan electric power system by placing as fewphase measurement units (PMUs) in the systemaspossible is closely relatedtothewell-known domination problem in graphs. In this work, we solve the power domination problem for WK-recursivenetworks.

Keywords:Powerdomination,WK-recursivenetwork,Sierpínskigraphs

I. INTRODUCTION

A dominating set of a graph $G(V, E)$ is a set S of vertices such that every vertex (node) in $V - S$ has at least one neighbour in S. The problem of finding a dominating set of minimum cardinality is an important problem that has been extensively studied. The minimum cardinality of a dominating set of G is its *domination number*, denoted by $\gamma(G)$. Our focus is on a variation called the power dominating set (PDS) problem.The power domination problem arose in the context of monitoring electric power networks. A power network contains a set of nodes and a set of edges connecting the nodes, also contains a set of generators, which supply power, and a set of loads, where the power is directed to. In order to monitor a power network we need to measure all the state variables of the network by placing measurement devices. A Phase Measurement Unit (PMU) is a measurement device placed on a node that has the ability to measure the voltage of the node and current phase of the edges connected to the node and to give warnings of system-wide failures. The goal is to install the minimum number of PMUs such that the whole system is monitored. This problem has been formulated as a graph domination problem by Haynes et al. in [1]. However, this type of domination is different from the standard domination type problem, since the domination rules can be iterated. The propagation rules are derived from the Ohm's and Kirchoff 's laws for an electric circuit.Let the graph $G(V, E)$ represent an electric power system, where a vertex represents an electrical component such as a PMU and an edge represents a transmission line joining two electrical nodes. A PMU measures the state variable for the vertex at which it is placed as well as its incident edges and their end vertices (these vertices and edges are said to be observed).

The other observation rules are as follows:
1. Any vertex that is incident to an observed edge is observed.

2. Any edge joining two observed vertices is observed.

3. If a vertex is incident to a total of $k > 1$ edges and if $k - 1$ of these edges are observed, then all k of these edges are observed.

Algorithmically, let G be a connected graph and S a subset of its vertices. Then we denote the set monitored by S with $M(S)$ and define it recursively as follows:
1. (domination)
$M(S) \leftarrow S \cup N(S)$
2. (propagation)

As long as there exists $v \in M(S)$ such that
$N(v) \cap (V(G) - M(S)) = \{w\}$
set $M(S) \leftarrow M(S) \cup \{w\}$.A set S is called a power dominating set (PDS) of Gif $M(S) = V(G)$. The power domination number $\gamma_p(G)$ is the minimum cardinality of a PDS of G. A PDS of G with the minimum cardinality is called a $\gamma_p(G)$-set. Since any dominating set is a power dominating set, $1 \leq \gamma_p(G) \leq \gamma(G)$ for all graphs G. $\gamma_p(G) = 1$in the case of cycle, path, complete graphs [1]. We say a graph G is power dominated by a set S if all its vertices are observed. For a vertex $v$of G, let $N(v)$ and $N[v]$ denote the open and closed neighbourhood of v respectively. For a set S, let $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = N(S) \cup S$ denote the open and close neighbourhood of S respectively. Let the notation $x \sim y$ mean that x is adjacent to y.The problem of deciding if a graph G has a power dominating set of cardinality k has been shown to be NP-complete even for bipartite graphs, chordal graphs [1] or even split graphs [2]. The power domination problem has efficient polynomial time algorithms for the classes of trees [1], graphs with bounded treewidth [3], block graphs [4, 5], block-cactus graphs [4], interval graphs [2], grids [6], honeycomb meshes [7] and circular-arc graphs [8]. Upper bounds on the power domination number are given for a connected graph with at least three vertices, for a connected claw-free cubic graph [9], for hypercubes [10], and for generalized Peterson graphs [11]. Closed formulas for the power domination number are obtained for Mycielskian of the complete graph, the wheel, the n-fan and n-star [12], for Cartesian product of paths and cycles [11, 13], for tensor and strong product of paths with paths [14], and for tensor product of paths with cycles [12].The next section deals with the power domination problem in WK - recursive networks.

II. WK-RECURSIVE STRUCTURES

ThearchitectureoftheWK-recursivenetworksdenotedby$WK(K, L)$[15]dependsontheequa edby$WK(K, L)$[15]dependsontheequality betweentheamplitude$W$andthedegree$K$ofvirtualnodesand Ltheexpansion-level.The firstlevelvirtualnodeis ing$K$realnodesof degree$K$toeachotherinafully connectedconfiguration,andleaving$K$linksfree.Therefore,avirtualnode is-virtually similartorealnodeofdegree$K$. ner,Kfirst levelvirtualnodesmaybeusedtoconstructasecond-

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levelvirtualnode,also ofdegree,and soon,untillevel$L$, whichmaybeconstructedfrom$K$,$(L-1)^{th}$levelvirtual-nodes.Amplitude WoftheL^{th}levelvirtualnodeis thenumber-ofits$(L-1)^{th}$levelvirtualnodes,havingofcourse$=K$.TheWK-recursivetopologiesareidentifiedessentially bythefollow-inganalytic relation$L=log_K N$where$N$ isthenumber ofreal-nodes, Kisthenode degreeandListheexpansionlevel.

IntheWK-recursive graph$WK(K,L)$, thereare$K$cornerreal-nodes ofdegree$(K-1)$.Therefore, theedgeconnectivi-ty,whichisthesmallest numberoflinksthatcanbedeletedinorder-todisconnectthegraph,isequal to$(K-1)$. The nodeconnectiv-ityofthegraphisthesmallestnumber of nodesthatcanbe-deletedinordertodisconnectthegraphandisalsoequal to$(K-1)$. The diameteroftheWK-recursivetopologies is$D=2^L-1$. Ingeneral,thediameterdepends onlyontheexpan-sion levelwhateverthe nodedegreeis.

$WK(K,L)$hasK^L verticesand $(K^{L+1}-K)/2$edges. $WK(K,L)$isa recursivestructure.Itconsistsof$K$copie-sof$WK(K,L-1)$orK^2copies of $WK(K,L-2)$ and so on. Thus, $WK(K,L)$ contains K^{L-1}copies of $WK(K,1)$. Since W K (K, 1) is a complete graph on K vertices. The following result is obvious.

Theorem 2.1.*Let G be* $WK(K,1)$. *Then* $\gamma_p(G)=1$.

Remark 2.2.*A complete graph in* $WK(K,L)$*can be power dominated if and only if*

1. *at least one vertex of the complete graph belongs to the power dominating set or*
2. *at least* $K-1$*vertices of the complete graph are observed.*

We shall now state a result that would lead us to the lower bound.

Theorem 2.3.*At least* $K-2$*vertices of each copy of* $WK(K,2)$*in* $WK(K,L)$*should belong to any power dominating set.*

*Proof.*Assume the contrary. Let $WK(K,L)$ be observed by taking $K-3$ vertices of a copy say W_1 of $WK(K,2)$ in a power dominating set D. The sub-graphW_1has$K$copiesof$WK(K,1)$. Letusdenotethesetof$K-3$ver-tice-sofW_1in$D$as$X=\{v_1,v_2,\ldots,v_{K-3}\}$.Then, $|N(D-X)\cap V(W_1)|\leq K$.Thereare${}^{K^2}C_{K-3}$choiceso-felementsof$XinW_1$.Butitisenoughtodiscuss${}^{K}C_{K-3}$vari-ouspossibilitiesaschoosingonevertexfromacopyof $W(K,1)$in$D$ hasthesameeffectaschoosingany othervertexfromthat copy.Allthesepossibilitiescanbeclubbedastwocases:

Case1: Everycopyof$WK(K,1)$inW_1hasatmostonevertexin$D$.

Case2:Atleastonecopyof$WK(K,1)$inW_1hastwoormorevertices in$D$.

Since$|X|=K-3$,thereareatleastthreecopiesof$WK(K,1)$in W_1thatdoesnothaveavertexin$D$.
Then,thesecopiesof$W(K,1)$willhaveat leasttwoverticesthata-renotobservedas$|N(D-X)\cap V(W_1)|\leq K$and vertex-in$D$chosenfromacopyof$W(K,1)inW_1$observesallverticesfrom thecopyandexactlyonevertexfromtheremainingcopie-sof$WK(K,1)$inW_1.
Inboththecas-es,W_1isnotobservedandhencethegraph$WK(K,L)$,contradictiont otheassumption.

SincethereareK^{L-2}copiesof$WK(K,1)$in$WK(K,L)$, thefollowing resultfollowsfromTheorem 2.3

Theorem2.4.*LetGbeWK-recursivenetwork. Then* $\gamma_p(G)\geq(K-2)\times K^{L-2}$

Inordertoshowthattheboundob-tainedissharp,weconstructapower dominatingsetofcardinality $(K-2)\times K^{L-2}$.The followingobservations arekeyinprov-ingtheupperbound.

Remark2.5.*ThevertexunionofallcopiesofWK(K,i),1≤i≤Lequals thevertexsetofWK(K,L). WK(K,L)isobservedifeach copyofWK(K,i)isobservedindependently*.

Sincethelowerboundwasob-tainedfromWK(K,2),wenextconcentrate onwhatconditionsdo-copiesofWK(K,2)inWK(K,L)getobserved. This leadsustothe-followingresult.

Theorem2.6.*ForanyPDSDandanycopy*W_1 *ofWK(K,2),*$V(W_1)$*is observedif*

1.$|V(W_1)\cap D|\geq K-1$*or*

2.$|V(W_1)\cap D|=K-2$*and*$|N(D-X)\cap V(W_1)|=\varphi$ *where*$X=\{v\in V(W_1)|v\in D\}$

*Proof.*Letusassumethat$|V(W_1)\cap D|\geq K-1$.

Case1:When$|V(W_1)\cap D|=K$, place K verticesoneeachin Kcopieso-f$WK(K,1)$inW_1.Then,byRemark2.2,eachcopyof$WK(K,1)$isobser vedandhenceW_1.

Case2:When $|V(W_1)\cap D|=K-1$,place$K-1$verticeso-neeachin$K-1$-copiesof$WK(K,1)$inW_1. Then byTheorem 2.2,$K-1$copiesof$WK(K,1)$areobserved.
W_1hasonecopyof$WK(K,1)$(sayW_2) thathas $K-1$ofitsverti-cesobserved. Bycondition2ofRemark 2.2,W_2 willalso beob-servedandhenceW_1.

Toprovethesecondstate-ment,letusassume$|V(W_1)\cap D|=K-2$.Inthiscase,place $K-2$verticesoneeachin$K-1$-copiesof$WK(K,1)$inW_1.ThenbyTheorem2.2,$K-2$copiesof
sof$WK(K,1)$inW_1.ThenbyTheorem2.2,$K-2$copiesof

$WK(K,1)$areobserved. W_1hastwocopiesof$WK(K,1)$(say$W_2$and W_3)with twounobserved verticeseach. When $|N(D-X)\cap V(W_1)| = \varphi$where $X = \{v \in V(W_1) | v \in D\}$,thenby Remark2.2,$W_2andW_3$willbeobservedandhencethegraphW_1.

Corollory:$\gamma_p(WK(K,2)) = K-1$.

Tolocateverticesof$WK(K,L)$inthepowerdominatingset,welabeltheverticesof$WK(K,L)$ as in [17]

We observe that the above algorithm is proper since each vertex in W K (K, L) receives a unique label, as at every stage of the algorithm only the unlabelled vertices are labelled.

Theorem 2.7.*The set* $D = \{1 + p.K(K+1) + jK, K^2 - Ki\} + mK^3$,$0 \le m \le K^{L-3} - 1, 0 \le p \le K-2$*for all* $m, 0 \le i,j \le K-3$*for all* $m$and$p$*, is a power dominating set for* $WK(K,L), K \ge 4, L \ge 3$.

*Proof.*Let us prove the result by the method of induction on L.

Base Case: L = 3.

This case reduces D for the graph *W K (K, 3)*. Let us denote K copies of *W K (K, 2)* in *WK (K, 3)* as $W_0, W_1, \ldots, W_{K-1}$ taken in the anti-clockwise sense.

In $WK(K,3)$, $D = \{1 + p.K(K+1) + jK, K^2 - Ki\}$, $0 \le p \le K-2$,

$0 \le i,j \le K-3$for all p.

When $p = 0$and $0 \le j \le K-3$, $D \supseteq X_0 = \{1 + jk\} \in V(W_0)$.

When $p = 1$and $0 \le j \le K-3$, $D \supseteq X_1 = \{1 + K(K+1) + jK\} \in V(W_1)$.

When $p = 2$and $0 \le j \le K-3$, $D \supseteq X_2 = \{1 + 2K(K+1) + jK\} \in V(W_2)$.

Similarly, When $p = K-2$and $0 \le j \le K-3$,

$D \supseteq X_{K-2} = \{1 + (K-2)K(K+1) + jK\} \in V(W_{K-2})$

and $D \supseteq X_{K-1} = \{K^2 - Kj\} \in V(W_{K-1})$.In all these cases $|Xi| = K-2$where $Xi = D \cap V(Wi)$. Also, the vertex $K^2 + 1 + j(K^2 + K + 1)$ of W_{j+1} is observed by the vertex $K + 1 + j(K^2 + K + 1)$ of W_j in *WK (K, 3)* and thus $|N(D - Xi) \cap V(W1)| = \varphi$ □ where $Xi = \{v \in V(W1) | v \in D\}$. Thus, by Theorem 2.6, all W_i are observed and hence *W K (K, 3)*.

Case: L = 4. *W K (K, 4)*has three copies of *W K (K, 3)*. Locate the same set of three points in three copies of *W K (K, 3)* i.e., $D = \{1 + p.K(K+1) + jK, K^2 - Ki\} + mK^3$, $0 \le m \le K^{L-3} - 1, 0 \le p \le K-2$ for all m, $0 \le j \le K-3$ for all m and p. By induction, the set D is obviously a power dominating set as each copy of $WK(K,3)$ is independently dominated.

Let us assume that the result is true for $L = k$. We shall prove for$L = k+1$. The graph $WK(K,k+1)$ has three copies of $WK(K,k)$. By induction, each copy is independently resolved as we locate the same set of points in each subgraph and hence the graph $WK(K,k+1)$ is observed.

Theorem 2.8.*The set* $D = \{1 + j.3^3, 10 + j.3^3, 23 + j.3^3\}$*where* $0 \le j \le$

$3^{L-3} - 1$*is a power dominating set for* $WK(3,L), L \ge 3$*and has cardinality*

3^{L-2}. *That is,* γ_p(W K (K, 3)) ≤ 3^{L-2}.

*Proof.*The proof is by induction on *L*. When *L = 3*, consider the set $D =$

$\{1, 10, 23\}$in $WK(3,3)$.. The set D satisfies the conditions of Theorem 2.6 and hence is a power dominating set.

When $L = 4$, $WK(3,4)$ has three copies of $WK(3,3)$. Locate the same set of three points in three copies of $WK(3,3)$ i.e., $D = \{1, 10, 23, 28, 37, 50, 55, 64, 77\}$. The set D is obviously a power dominating set as each copy of $WK(3,3)$ is independently dominated. Algorithm this procedure to obtain the power dominating set for any L.

Let us assume that the result is true for $L = k$. We shall prove for $L = k + 1$. The graph$WK(3,k+1)$ has three copies of $WK(3,k)$. By induction, each copy is independently resolved as we locate the same set of points in each subgraph and hence the graph $WK(3,k+1)$ is observed.

By Theorem 2.4, 2.7 and 2.8, we now state

Theorem 2.9.*Let* G *be* W K - *recursive network*, W K (K, L), K ≥ 3, L ≥ 3. *Then*

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$$\gamma_p(G) = (K - 2) \times K^{L-2}$$

It is to be noted that $W K (3, L)$ is Sierp´ınski graph of dimension L [16]

Theorem 2.10.*Let* G *be a Sierp´ınski graph of dimension* $n, n \geq 2$. *Then* $\gamma_p(G) = 3^{n-2}$

III. CONCLUSION

In this paper, power domination problem is solved for W K - recursive networks and Sierp´ınski graph. The result has motivated us to believe that subgraphs have a certain role in determining the power domination of a graph. In our future research, we intend to focus on the relationship between the two.

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